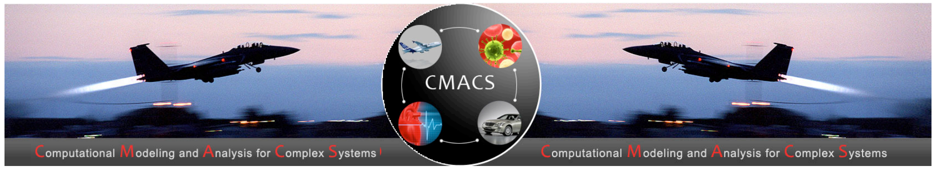
NSF CMACS expedition, PI meeting, Pittsburgh, 04/28–29/2011



Unifying proof theoretic/logical and algebraic abstractions for inference and verification

Patrick Cousot
NYU
pcousot@cs.nyu.edu cs.nyu.edu/~pcousot

Objective

Algebraic abstractions

- Used in abstract interpretation, model-checking,...
- System properties and specifications are abstracted as an algebraic lattice (abstraction-specific encoding of properties)
- Fully automatic: system properties are computed as fixpoints of algebraic transformers
- Several separate abstractions can be combined with the reduced product

Proof theoretic/logical abstractions

- Used in deductive methods
- System properties and specifications are expressed with formulæ of first-order theories (universal encoding of properties)
- Partly automatic: system properties are provided manually by end-users and automatically checked to satisfy verification conditions (with implication defined by the theories)
- Various theories can be combined by Nelson-Oppen procedure

Objective

- Show that proof-theoretic/logical abstractions are a particular case of algebraic abstractions
- Show that Nelson-Oppen procedure is a particular case of reduced product
- Use this unifying point of view to propose a new combination of logical and algebraic abstractions

➥ Convergence of proof theoretic/ logical and algebraic property-inference and verification methods

Concrete semantics

Programs (syntax)

- Expressions (on a signature $\langle \mathbb{f}, \mathbb{p} \rangle$)

$\mathtt{x}, \mathtt{y}, \mathtt{z}, \ldots \in \mathbb{x}$		variables
$\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots \in \mathbb{f}^0$		constants
$\mathtt{f}, \mathtt{g}, \mathtt{h}, \ldots \in \mathbb{f}^n, \quad \mathbb{f} \triangleq \bigcup_{n \geqslant 0} \mathbb{f}^n$		function symbols of arity $n \geqslant 1$
$t \in \mathbb{T}(\mathbb{x}, \mathbb{f})$	$t ::= \mathtt{x} \mid \mathtt{c} \mid \mathtt{f}(t_1, \ldots, t_n)$	terms
$\mathtt{p}, \mathtt{q}, \mathtt{r}, \ldots \in \mathbb{p}^n, \quad \mathbb{p}^0 \triangleq \{\mathtt{ff}, \mathtt{tt}\}, \quad \mathbb{p} \triangleq \bigcup_{n \geqslant 0} \mathbb{p}^n$		predicate symbols of arity $n \geqslant 0$,
$a \in \mathbb{A}(\mathbb{x}, \mathbb{f}, \mathbb{p})$	$a ::= \mathtt{ff} \mid \mathtt{p}(t_1, \ldots, t_n) \mid \neg a$	atomic formulæ
$e \in \mathbb{E}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \triangleq \mathbb{T}(\mathbb{x}, \mathbb{f}) \cup \mathbb{A}(\mathbb{x}, \mathbb{f}, \mathbb{p})$		program expressions
$\varphi \in \mathbb{C}(\mathbb{x}, \mathbb{f}, \mathbb{p})$	$\varphi ::= a \mid \varphi \wedge \varphi$	clauses in simple conjunctive normal form

- Programs (including assignment, guards, loops, ...)

$\mathtt{P}, \ldots \in \mathbb{P}(\mathbb{x}, \mathbb{f}, \mathbb{p})$	$\mathtt{P} ::= \mathtt{x} := e \mid \varphi \mid \ldots$	programs

Programs (interpretation)

- Interpretation $I \in \mathfrak{I}$ for a signature $\langle \mathbb{f}, \mathbb{p} \rangle$ is $\langle I_{\mathcal{V}}, I_\gamma \rangle$ such that

— $I_{\mathcal{V}}$ is a non-empty set of values,
— $\forall \mathtt{c} \in \mathbb{f}^0 : I_\gamma(\mathtt{c}) \in I_{\mathcal{V}}, \quad \forall n \geqslant 1 : \forall \mathtt{f} \in \mathbb{f}^n : I_\gamma(\mathtt{f}) \in I_{\mathcal{V}}^n \rightarrow I_{\mathcal{V}}$,
— $\forall n \geqslant 0 : \forall \mathtt{p} \in \mathbb{p}^n : I_\gamma(\mathtt{p}) \in I_{\mathcal{V}}^n \rightarrow \mathcal{B}$. $\quad \mathcal{B} \triangleq \{\mathit{false}, \mathit{true}\}$

- Environments

$$\eta \in \mathcal{R}_I \triangleq \mathbb{x} \rightarrow I_{\mathcal{V}} \quad \text{environments}$$

- Expression evaluation

$[\![a]\!]_I \eta \in \mathcal{B}$ of an atomic formula $a \in \mathbb{A}(\mathbb{x}, \mathbb{f}, \mathbb{p})$
$[\![t]\!]_I \eta \in I_{\mathcal{V}}$ of the term $t \in \mathbb{T}(\mathbb{x}, \mathbb{f})$

Programs (concrete semantics)

- The program semantics is usually specified relative to a standard interpretation $\mathfrak{I} \in \mathbb{I}$.
- The concrete semantics is given in post-fixpoint form (in case the least fixpoint which is also the least post-fixpoint does not exist, e.g. *inexpressibility* in Hoare logic)

$\mathcal{R}_{\mathfrak{I}}$	concrete observables[5]
$\mathcal{P}_{\mathfrak{I}} \triangleq \wp(\mathcal{R}_{\mathfrak{I}})$	concrete properties [6]
$F_{\mathfrak{I}}[\![\mathtt{P}]\!] \in \mathcal{P}_{\mathfrak{I}} \to \mathcal{P}_{\mathfrak{I}}$	concrete transformer of program P
$C_{\mathfrak{I}}[\![\mathtt{P}]\!] \triangleq \mathbf{postfp}^{\subseteq} F_{\mathfrak{I}}[\![\mathtt{P}]\!] \in \wp(\mathcal{P}_{\mathfrak{I}})$	concrete semantics of program P

where $\mathbf{postfp}^{\leq} f \triangleq \{x \mid f(x) \leq x\}$

[5] Examples of observables are set of states, set of partial or complete execution traces, infinite/transfinite execution trees, etc.
[6] A property is understood as the set of elements satisfying this property.

Example of program concrete semantics

- Program $\mathtt{P} \triangleq$ `x=1; while true {x=incr(x)}`
- Arithmetic interpretation $\mathfrak{I}$ on integers $\mathfrak{I}_{\mathcal{V}} = \mathbb{Z}$
- Loop invariant $\mathbf{lfp}^{\subseteq} F_{\mathfrak{I}}[\![\mathtt{P}]\!] = \{\eta \in \mathcal{R}_{\mathfrak{I}} \mid 0 < \eta(\mathtt{x})\}$

  where
  $\mathcal{R}_{\mathfrak{I}} \triangleq \mathbb{x} \to \mathfrak{I}_{\mathcal{V}}$ concrete environments
  $F_{\mathfrak{I}}[\![\mathtt{P}]\!](X) \triangleq \{\eta \in \mathcal{R}_{\mathfrak{I}} \mid \eta(\mathtt{x}) = 1\} \cup \{\eta[\mathtt{x} \leftarrow \eta(\mathtt{x}) + 1] \mid \eta \in X\}$
- The *strongest invariant* is $\mathbf{lfp}^{\subseteq} F_{\mathfrak{I}}[\![\mathtt{P}]\!] = \bigcap \mathbf{postfp}^{\subseteq} F_{\mathfrak{I}}[\![\mathtt{P}]\!]$
- *Expressivity*: the $\mathbf{lfp}$ may not be expressible in the abstract in which case we use the set of possible invariants $C_{\mathfrak{I}}[\![\mathtt{P}]\!] \triangleq \mathbf{postfp}^{\subseteq} F_{\mathfrak{I}}[\![\mathtt{P}]\!]$

Concrete domains

- The standard semantics describes computations of a system formalized by elements of a domain of observables $\mathcal{R}_{\mathfrak{I}}$ (e.g., set of traces, states, etc)

  The properties $\mathcal{P}_{\mathfrak{I}} \triangleq \wp(\mathcal{R}_{\mathfrak{I}})$ (a property is the set of elements with that property) form a complete lattice $\langle \mathcal{P}_{\mathfrak{I}}, \subseteq, \emptyset, \mathcal{R}_{\mathfrak{I}}, \cup, \cap \rangle$
- The concrete semantics $C_{\mathfrak{I}}[\![\mathtt{P}]\!] \triangleq \mathbf{postfp}^{\subseteq} F_{\mathfrak{I}}[\![\mathtt{P}]\!]$ defines the system properties of interest for the verification
- The transformer $F_{\mathfrak{I}}[\![\mathtt{P}]\!]$ is defined in terms of primitives, e.g.

$\mathrm{f}_{\mathfrak{I}}[\![\mathtt{x} := e]\!]P \triangleq \{\eta[\mathtt{x} \leftarrow [\![e]\!]_{\mathfrak{I}}\eta] \mid \eta \in P\}$	Floyd's assignment post-condition
$\mathrm{p}_{\mathfrak{I}}[\![\varphi]\!]P \triangleq \{\eta \in P \mid [\![\varphi]\!]_{\mathfrak{I}}\eta = true\}$	test

Extension to multi-interpretations

- Programs have many interpretations $\mathcal{I} \in \wp(\mathbb{I})$.
- Multi-interpreted semantics

$\mathcal{R}_I$	program observables for interpretation $I \in \mathcal{I}$
$\mathcal{P}_{\mathcal{I}} \triangleq I \in \mathcal{I} \not\mapsto \wp(\mathcal{R}_I)$ $\simeq \wp(\{\langle I, \eta\rangle \mid I \in \mathcal{I} \wedge \eta \in \mathcal{R}_I\})$[8]	interpreted properties for the set of interpretations $\mathcal{I}$

$F_{\mathcal{I}}[\![\mathtt{P}]\!] \in \mathcal{P}_{\mathcal{I}} \to \mathcal{P}_{\mathcal{I}}$ $\triangleq \lambda P \in \mathcal{P}_{\mathcal{I}} \cdot \lambda I \in \mathcal{I} \cdot F_I[\![\mathtt{P}]\!](P(I))$	multi-interpreted concrete transformer of program P
$C_{\mathcal{I}}[\![\mathtt{P}]\!] \in \wp(\mathcal{P}_{\mathcal{I}})$ $\triangleq \mathbf{postfp}^{\dot{\subseteq}} F_{\mathcal{I}}[\![\mathtt{P}]\!]$	multi-interpreted concrete semantics

where $\dot{\subseteq}$ is the pointwise subset ordering.

[8] A partial function $f \in A \nrightarrow B$ with domain $\mathsf{dom}(f) \in \wp(A)$ is understood as the relation $\{\langle x, f(x)\rangle \in A \times B \mid x \in \mathsf{dom}(f)\}$ and maps $x \in A$ to $f(x) \in B$, written $x \in A \not\mapsto f(x) \in B$ or $x \in A \not\mapsto B_x$ when $\forall x \in A : f(x) \in B_s \subseteq B$.

Algebraic Abstractions

Abstract domains

$\langle A, \sqsubseteq, \bot, \top, \sqcup, \sqcap, \nabla, \Delta, \bar{\mathsf{f}}, \bar{\mathsf{b}}, \bar{\mathsf{p}}, \ldots \rangle$

where

$\overline{P}, \overline{Q}, \ldots \in A$	abstract properties
$\sqsubseteq \; \in A \times A \to \mathcal{B}$	abstract partial order [9]
$\bot, \top \in A$	infimum, supremum
$\sqcup, \sqcap, \nabla, \Delta \in A \times A \to A$	abstract join, meet, widening, narrowing
$\ldots$	
$\bar{\mathsf{f}} \in (\mathbb{x} \times \mathbb{E}(\mathbb{x}, \mathbb{f}, \mathbb{p})) \to A \to A$	abstract forward assignment transformer
$\bar{\mathsf{b}} \in (\mathbb{x} \times \mathbb{E}(\mathbb{x}, \mathbb{f}, \mathbb{p})) \to A \to A$	abstract backward assignment transformer
$\bar{\mathsf{p}} \in \mathbb{C}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \to A \to A$	abstract condition transformer.

Abstract semantics

- A abstract domain
- $\sqsubseteq$ abstract logical implication
- $\overline{F}[\![\mathrm{P}]\!] \in A \to A$ abstract transformer defined in term of abstract primitives

$\bar{\mathsf{f}} \in (\mathbb{x} \times \mathbb{E}(\mathbb{x}, \mathbb{f}, \mathbb{p})) \to A \to A$	abstract forward assignment transformer
$\bar{\mathsf{b}} \in (\mathbb{x} \times \mathbb{E}(\mathbb{x}, \mathbb{f}, \mathbb{p})) \to A \to A$	abstract backward assignment transformer
$\bar{\mathsf{p}} \in \mathbb{C}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \to A \to A$	abstract condition transformer.

- $\overline{C}[\![\mathrm{P}]\!] \triangleq \{\mathbf{lfp}^{\sqsubseteq} \overline{F}[\![\mathrm{P}]\!]\}$ least fixpoint semantics, if any
- $\overline{C}[\![\mathrm{P}]\!] \triangleq \{\overline{P} \mid \overline{F}[\![\mathrm{P}]\!](\overline{P}) \sqsubseteq \overline{P}\}$ or else, post-fixpoint abstract semantics

Soundness of the abstract semantics

- Concretization $\gamma \in A \stackrel{\nearrow}{\longrightarrow} \mathcal{P}_{\mathfrak{I}}$
- Soundness of the abstract semantics
  $\forall \overline{P} \in A : (\exists \overline{C} \in \overline{C}[\![\mathrm{P}]\!] : \overline{C} \sqsubseteq \overline{P}) \Rightarrow (\exists C \in C[\![\mathrm{P}]\!] : C \subseteq \gamma(\overline{P}))$
- Sufficient local soundness conditions:

$(\overline{P} \sqsubseteq \overline{Q}) \Rightarrow (\gamma(\overline{P}) \subseteq \gamma(\overline{Q}))$	order	$\gamma(\bot) = \emptyset$	infimum
$\gamma(\overline{P} \sqcup \overline{Q}) \supseteq (\gamma(\overline{P}) \cup \gamma(\overline{Q}))$	join	$\gamma(\top) = \top_{\mathfrak{I}}$	supremum

...

$\gamma(\bar{\mathsf{f}}[\![\mathtt{x} := e]\!]\overline{P}) \supseteq \mathsf{f}_{\mathfrak{I}}[\![\mathtt{x} := e]\!]\gamma(\overline{P})$	assignment post-condition
$\gamma(\bar{\mathsf{b}}[\![\mathtt{x} := e]\!]\overline{P}) \supseteq \mathsf{b}_{\mathfrak{I}}[\![\mathtt{x} := e]\!]\gamma(\overline{P})$	assignment pre-condition
$\gamma(\bar{\mathsf{p}}[\![\varphi]\!]\overline{P}) \supseteq \mathsf{p}_{\mathfrak{I}}[\![\varphi]\!]\gamma(\overline{P})$	test

implying $\forall \overline{P} \in A : F[\![\mathrm{P}]\!] \circ \gamma(\overline{P}) \subseteq \gamma \circ \overline{F}[\![\mathrm{P}]\!](\overline{P})$

Beyond bounded verification: Widening

- Definition of widening:

Let $\langle A, \sqsubseteq\rangle$ *be a poset. Then an* over-approximating widening $\triangledown \in A \times A \mapsto A$ *is such that*

(a) $\forall x, y \in A : x \sqsubseteq x \triangledown y \wedge y \leqslant x \triangledown y$[14].

A terminating widening $\triangledown \in A \times A \mapsto A$ *is such that*

(b) *Given any sequence* $\langle x^n,\ n \geqslant 0\rangle$, *the sequence* $y^0 = x^0, \ldots, y^{n+1} = y^n \triangledown x^n, \ldots$ *converges (i.e.* $\exists \ell \in \mathbb{N} : \forall n \geqslant \ell : y^n = y^\ell$ *in which case* y^ℓ *is called the* limit *of the widened sequence* $\langle y^n,\ n \geqslant 0\rangle$*).*

Traditionally a widening *is considered to be both over-approximating and terminating.* □

Beyond bounded verification: Widening

- Iterations with widening

The iterates of a transformer $\overline{F}[\![P]\!] \in A \mapsto A$ *from the infimum* $\bot \in A$ *with widening* $\triangledown \in A \times A \mapsto A$ *in a poset* $\langle A, \sqsubseteq\rangle$ *are defined by recurrence as* $\overline{F}^0 = \bot$, $\overline{F}^{n+1} = \overline{F}^n$ *when* $\overline{F}[\![P]\!](\overline{F}^n) \sqsubseteq \overline{F}^n$ *and* $\overline{F}^{n+1} = \overline{F}^n \triangledown \overline{F}[\![P]\!](\overline{F}^n)$ *otherwise.* □

- Soundness of iterations with widening

The iterates in a poset $\langle A, \sqsubseteq, \bot\rangle$ *of a transformer* $\overline{F}[\![P]\!]$ *from the infimum* $\bot$ *with widening* $\triangledown$ *converge and their limit is a post-fixpoint of the transformer.* □

Implementation notes

- Each abstract domain $\langle A, \sqsubseteq, \bot, \top, \sqcup, \sqcap, \triangledown, \vartriangle, \bar{\mathsf{f}}, \bar{\mathsf{b}}, \bar{\mathsf{p}}, \ldots\rangle$ is implemented separately by hand, by providing a specific computer representation of properties in A, and algorithms for the logical operations $\sqsubseteq, \bot, \top, \sqcup, \sqcap$, and transformers $\bar{\mathsf{f}}, \bar{\mathsf{b}}, \bar{\mathsf{p}}, \ldots$
- Different abstract domains are combined into a reduced product
- Very efficient but implemented manually (requires skilled specialists)

First-order logic

First-order logical formulæ & satisfaction

- Syntax

$\Psi \in \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \qquad \Psi ::= a \mid \neg\Psi \mid \Psi \wedge \Psi \mid \exists \mathrm{x} : \Psi$ quantified first-order formulæ

a distinguished predicate $= (t_1, t_2)$ which we write $t_1 = t_2$

- Free variables $\vec{\mathrm{x}}_\Psi$
- Satisfaction

$I \models_\eta \Psi$, interpretation I and an environment η satisfy a formula Ψ

- Equality

$$I \models_\eta t_1 = t_2 \ \triangleq\ [\![t_1]\!]_I \eta =_I [\![t_2]\!]_I \eta$$

where $=_I$ is the unique reflexive, symmetric, antisymmetric, and transitive relation on $I_\mathcal{V}$.

Extension to multi-interpretations

- Property described by a formula for multiple interpretations

$$\mathcal{I} \in \wp(\mathfrak{I})$$

- Semantics of first-order formulæ

$$\gamma^{\mathfrak{a}}_{\mathcal{I}} \in \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \xrightarrow{\nearrow} \mathcal{P}_{\mathcal{I}}$$
$$\gamma^{\mathfrak{a}}_{\mathcal{I}}(\Psi) \triangleq \{\langle I, \eta\rangle \mid I \in \mathcal{I} \wedge I \models_\eta \Psi\}$$

- But how are we going to describe sets of interpretations $\mathcal{I} \in \wp(\mathfrak{I})$?

Defining multiple interpretations as models of theories

- Theory: set $\mathcal{T}$ of theorems (closed sentences without any free variable)
- Models of a theory (interpretations making true all theorems of the theory)

$$\begin{aligned}\mathfrak{M}(\mathcal{T}) &\triangleq \{I \in \mathfrak{I} \mid \forall \Psi \in \mathcal{T} : \exists \eta : I \models_\eta \Psi\}\\ &= \{I \in \mathfrak{I} \mid \forall \Psi \in \mathcal{T} : \forall \eta : I \models_\eta \Psi\}\end{aligned}$$

Classical properties of theories

- Decidable theories: $\forall \Psi \in \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}) : \mathsf{decide}_{\mathcal{T}}(\Psi) \triangleq (\Psi \in \mathcal{T})$ is computable
- Deductive theories: closed by deduction

$$\forall \Psi \in \mathcal{T} : \forall \Psi' \in \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}), \text{ if } \Psi \Rightarrow \Psi' \text{ implies } \Psi' \in \mathcal{T}$$

- Satisfiable theory:

$$\mathfrak{M}(\mathcal{T}) \neq \emptyset$$

- Complete theory:

for all sentences Ψ in the language of the theory, either Ψ is in the theory or $\neg\Psi$ is in the theory.

Checking satisfiability modulo theory

- Validity modulo theory

$$\text{valid}_{\mathcal{T}}(\Psi) \triangleq \forall I \in \mathfrak{M}(\mathcal{T}) : \forall \eta : I \models_\eta \Psi$$

- Satisfiability modulo theory (SMT)

$$\text{satisfiable}_{\mathcal{T}}(\Psi) \triangleq \exists I \in \mathfrak{M}(\mathcal{T}) : \exists \eta : I \models_\eta \Psi$$

- Checking satisfiability for decidable theories

$$\text{satisfiable}_{\mathcal{T}}(\Psi) \Leftrightarrow \neg(\text{decide}_{\mathcal{T}}(\forall \vec{x}_\Psi : \neg\Psi)) \quad \text{(when } \mathcal{T} \text{ is decidable and deductive)}$$

$$\text{satisfiable}_{\mathcal{T}}(\Psi) \Leftrightarrow (\text{decide}_{\mathcal{T}}(\exists \vec{x}_\Psi : \Psi)) \quad \text{(when } \mathcal{T} \text{ is decidable and complete)}$$

- Most SMT solvers support only quantifier-free formulæ

Logical abstract domains

- $\langle A, \mathcal{T}\rangle$: $A \in \wp(\mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}))$ abstract properties

  $\mathcal{T}$ theory of $\mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p})$

- Abstract domain $\langle A, \sqsubseteq, \mathrm{ff}, \mathrm{tt}, \vee, \wedge, \nabla, \Delta, \bar{f}_a, \bar{b}_a, \bar{p}_a, \ldots\rangle$
- Logical implication $(\Psi \sqsubseteq \Psi') \triangleq ((\forall \vec{x}_\Psi \cup \vec{x}_{\Psi'} : \Psi \Rightarrow \Psi') \in \mathcal{T})$
- A lattice but in general not complete
- The concretization is

$$\gamma^a_{\mathcal{T}}(\Psi) \triangleq \left\{\langle I, \eta\rangle \;\middle|\; I \in \mathfrak{M}(\mathcal{T}) \wedge I \models_\eta \Psi\right\}$$

Logical Abstractions

Logical abstract semantics

- Logical abstract semantics

$$\overline{C}^a[\![\mathrm{P}]\!] \triangleq \left\{\Psi \mid \overline{F}_a[\![\mathrm{P}]\!](\Psi) \sqsubseteq \Psi\right\}$$

- The logical abstract transformer $\overline{F}_a[\![\mathrm{P}]\!] \in A \rightarrow A$ is defined in terms of primitives

$\bar{\mathrm{f}}_a \in (\mathbb{x} \times \mathbb{T}(\mathbb{x}, \mathbb{f})) \rightarrow A \rightarrow A$	abstract forward assignment transformer
$\bar{\mathrm{b}}_a \in (\mathbb{x} \times \mathbb{T}(\mathbb{x}, \mathbb{f})) \rightarrow A \rightarrow A$	abstract backward assignment transformer
$\bar{\mathrm{p}}_a \in \mathbb{L} \rightarrow A \rightarrow A$	condition abstract transformer

Implementation notes ...

- Universal representation of abstract properties by logical formulæ
- Trival implementations of logical operations $\mathbb{ff}, \mathbb{tt}, \vee, \wedge,$
- Provers or SMT solvers can be used for the abstract implication $\sqsubseteq$,
- Concrete transformers are purely syntactic

$\mathsf{f}_a \in (\mathbb{x} \times \mathbb{T}(\mathbb{x}, \mathbb{f})) \to \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \to \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p})$ $\mathsf{f}_a[\![\mathtt{x} := t]\!]\Psi \triangleq \exists x' : \Psi[\mathtt{x} \leftarrow x'] \wedge \mathtt{x} = t[\mathtt{x} \leftarrow x']$	axiomatic forward assignment transformer
$\mathsf{b}_a \in (\mathbb{x} \times \mathbb{T}(\mathbb{x}, \mathbb{f})) \to \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \to \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p})$ $\mathsf{b}_a[\![\mathtt{x} := t]\!]\Psi \triangleq \Psi[\mathtt{x} \leftarrow t]$	axiomatic backward assignment transformer
$\mathsf{p}_a \in \mathbb{C}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \to \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \to \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p})$ $\mathsf{p}_a[\![\varphi]\!]\Psi \triangleq \Psi \wedge \varphi$	axiomatic transformer for program test of condition φ.

.../...

but ...

.../... so the abstract transformers follows by abstraction

$\bar{\mathsf{f}}_a[\![\mathtt{x} := t]\!]\Psi \triangleq \alpha_A^{\mathcal{I}}(\mathsf{f}_a[\![\mathtt{x} := t]\!]\Psi)$	abstract forward assignment transformer
$\bar{\mathsf{b}}_a[\![\mathtt{x} := t]\!]\Psi \triangleq \alpha_A^{\mathcal{I}}(\mathsf{b}_a[\![\mathtt{x} := t]\!]\Psi)$	abstract backward assignment transformer
$\bar{\mathsf{p}}_a[\![\varphi]\!]\Psi \triangleq \alpha_A^{\mathcal{I}}(\mathsf{p}_a[\![\varphi]\!]\Psi)$	abstract transformer for program test of condition

- The abstraction algorithm $\alpha_A^{\mathcal{I}} \in \mathbb{F}(\mathbb{x}, \mathbb{f}, \mathbb{p}) \to A$ to abstract properties in A may be non-trivial (e.g. quantifiers elimination)
- A widening ∇ is needed to ensure convergence of the fixpoint iterates (or else ask the end-user)

Example I of widening: thresholds

- Choose a subset W of A satisfying the ascending chain condition for $\sqsubseteq$,
- Define $X \nabla Y$ to be (one of) the strongest $\Psi \in W$ such that $Y \Rightarrow \Psi$

Example II of bounded widening: Craig interpolation

- Use Craig interpolation (knowing a bound e.g. the specification)
- Move to thresholds to enforced convergence after k widenings with Craig interpolation

Reduced Product

Cartesian product

- Definition of the Cartesian product:

> *Let* $\langle A_i, \sqsubseteq_i\rangle$, $i \in \Delta$, Δ *finite, be abstract domains with increasing concretization* $\gamma_i \in A_i \xrightarrow{\nearrow} \mathfrak{P}^{\Sigma_O}_{I}$. *Their* Cartesian product *is* $\langle \vec{A}, \overrightarrow{\sqsubseteq}\rangle$ *where* $\vec{A} \triangleq \bigtimes_{i\in\Delta} A_i$, $(\vec{P} \mathrel{\overrightarrow{\sqsubseteq}} \vec{Q}) \triangleq \bigwedge_{i\in\Delta}(\vec{P}_i \sqsubseteq_i \vec{Q}_i)$ *and* $\vec{\gamma} \in \vec{A} \to \mathfrak{P}^{\Sigma_O}_{I}$ *is* $\vec{\gamma}(\vec{P}) \triangleq \bigcap_{i\in\Delta} \gamma_i(\vec{P}_i)$.

Reduced product

- Definition of the Reduced product:

> *Let* $\langle A_i, \sqsubseteq_i\rangle$, $i \in \Delta$, Δ *finite, be abstract domains with increasing concretization* $\gamma_i \in A_i \xrightarrow{\nearrow} \mathfrak{P}^{\Sigma_O}_{I}$ *where* $\vec{A} \triangleq \bigtimes_{i\in\Delta} A_i$ *is their Cartesian product. Their* reduced product *is* $\langle \vec{A}/_{\vec{\equiv}}, \overrightarrow{\sqsubseteq}\rangle$ *where* $(\vec{P} \mathrel{\vec{\equiv}} \vec{Q}) \triangleq (\vec{\gamma}(\vec{P}) = \vec{\gamma}(\vec{Q}))$ *and* $\vec{\gamma}$ *as well as* $\overrightarrow{\sqsubseteq}$ *are naturally extended to the equivalence classes* $[\vec{P}]/_{\vec{\equiv}}$, $\vec{P} \in \vec{A}$, *of* $\vec{\equiv}$ *by* $\vec{\gamma}([\vec{P}]/_{\vec{\equiv}}) = \vec{\gamma}(\vec{P})$ *and* $[\vec{P}]/_{\vec{\equiv}} \mathrel{\overrightarrow{\sqsubseteq}} [\vec{Q}]/_{\vec{\equiv}} \triangleq \exists \vec{P}' \in [\vec{P}]/_{\vec{\equiv}} : \exists \vec{Q}' \in [\vec{Q}]/_{\vec{\equiv}} : \vec{P}' \mathrel{\overrightarrow{\sqsubseteq}} \vec{Q}'$. $\square$

- In practice, the reduced product may be complex to compute but we can use approximations such as the iterated pairwise reduction of the Cartesian product

Reduction

- Example: intervals x congruences

  $\rho(\, x \in [-1,5] \wedge x = 2 \bmod 4) \equiv x \in [2,2] \wedge x = 2 \bmod 0$

  are equivalent

- Meaning-preserving reduction:

> *Let* $\langle A, \sqsubseteq\rangle$ *be a poset which is an abstract domain with concretization* $\gamma \in A \xrightarrow{\nearrow} C$ *where* $\langle C, \leqslant\rangle$ *is the concrete domain. A* meaning-preserving map *is* $\rho \in A \to A$ *such that* $\forall \overline{P} \in A : \gamma(\rho(\overline{P})) = \gamma(\overline{P})$. *The map is a* reduction *if and only if it is reductive that is* $\forall \overline{P} \in A : \rho(\overline{P}) \sqsubseteq \overline{P}$. $\square$

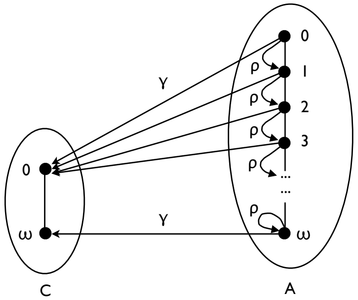
Iterated reduction

- Definition of iterated reduction:

> *Let* $\langle A, \sqsubseteq\rangle$ *be a poset which is an abstract domain with concretization* $\gamma \in A \xrightarrow{\nearrow} C$ *where* $\langle C, \subseteq\rangle$ *is the concrete domain and* $\rho \in A \to A$ *be a meaning-preserving reduction.*
>
> The iterates *of the reduction are* $\rho^0 \triangleq \lambda \overline{P} \bullet \overline{P}$, $\rho^{\lambda+1} = \rho(\rho^{\lambda})$ *for successor ordinals and* $\rho^{\lambda} = \bigsqcap_{\beta<\lambda} \rho^{\beta}$ *for limit ordinals.*
>
> *The iterates are* well-defined *when the greatest lower bounds* $\bigsqcap$ *(glb) do exist in the poset* $\langle A, \sqsubseteq\rangle$. $\square$

Finite versus infinite iterated reduction

- Finite iterations of a meaning preserving reduction are meaning preserving (and more precise)
- Infinite iterations, limits of meaning-preserving reduction, may not be meaning-preserving (although more precise). It is when γ preserves glbs.



Pairwise reduction

- Definition of pairwise reduction

Let $\langle A_i, \sqsubseteq_i\rangle$ be abstract domains with increasing concretization $\gamma_i \in A_i \xrightarrow{\nearrow} L$ into the concrete domain $\langle L, \leqslant\rangle$.

For $i, j \in \Delta$, $i \neq j$, let $\rho_{ij} \in \langle A_i \times A_j, \sqsubseteq_{ij}\rangle \mapsto \langle A_i \times A_j, \sqsubseteq_{ij}\rangle$ be pairwise meaning-preserving reductions (so that $\forall \langle x, y\rangle \in A_i \times A_j : \rho_{ij}(\langle x, y\rangle) \sqsubseteq_{ij} \langle x, y\rangle$ and $(\gamma_i \times \gamma_j) \circ \rho_{ij} = (\gamma_i \times \gamma_j)$[24]).

Define the pairwise reductions $\vec{\rho}_{ij} \in \langle \vec{A}, \vec{\sqsubseteq}\rangle \mapsto \langle \vec{A}, \vec{\sqsubseteq}\rangle$ of the Cartesian product as

$$\vec{\rho}_{ij}(\vec{P}) \triangleq \text{let } \langle \vec{P}'_i, \vec{P}'_j\rangle \triangleq \rho_{ij}(\langle \vec{P}_i, \vec{P}_j\rangle) \text{ in } \vec{P}[i \leftarrow \vec{P}'_i][j \leftarrow \vec{P}'_j]$$

where $\vec{P}[i \leftarrow x]_i = x$ and $\vec{P}[i \leftarrow x]_j = \vec{P}_j$ when $i \neq j$.

[24] We define $(f \times g)(\langle x, y\rangle) \triangleq \langle f(x), g(y)\rangle$.

Pairwise reduction (cont'd)

Define the iterated pairwise reductions $\vec{\rho}^{\,n}, \vec{\rho}^{\,\lambda}, \vec{\rho}^{\,*} \in \langle \vec{A}, \vec{\sqsubseteq}\rangle \mapsto \langle \vec{A}, \vec{\sqsubseteq}\rangle$, $n \geqslant 0$ of the Cartesian product for

$$\vec{\rho} \triangleq \bigcirc_{\substack{i,j \in \Delta,\\ i \neq j}} \vec{\rho}_{ij}$$

where $\bigcirc_{i=1}^{n} f_i \triangleq f_{\pi_1} \circ \ldots \circ f_{\pi_n}$ is the function composition for some arbitrary permutation π of $[1, n]$. □

Iterated pairwise reduction

- The iterated pairwise reduction of the Cartesian product is meaning preserving

If the limit $\vec{\rho}^{\,*}$ of the iterated reductions is well defined then the reductions are such that $\forall \vec{P} \in \vec{A} : \forall n \in \mathbb{N}_+ : \vec{\rho}^{\,\star}(\vec{P}) \vec{\sqsubseteq} \vec{\rho}^{\,n}(\vec{P}) \vec{\sqsubseteq} \vec{\rho}_{ij}(\vec{P}) \vec{\sqsubseteq} \vec{P}$, $i, j \in \Delta$, $i \neq j$ and meaning-preserving since $\vec{\rho}^{\,\lambda}(\vec{P}), \vec{\rho}_{ij}(\vec{P}), \vec{P} \in [\vec{P}]/_{\vec{\equiv}}$.

If, moreover, γ preserves greatest lower bounds then $\vec{\rho}^{\,\star}(\vec{P}) \in [\vec{P}]/_{\vec{\equiv}}$. □

Iterated pairwise reduction

- In general, the iterated pairwise reduction of the Cartesian product is not as precise as the reduced product
- Sufficient conditions do exist for their equivalence

Counter-example

- $L = \wp(\{a, b, c\})$
- $A_1 = \{\emptyset, \{a\}, \top\}$ where $\top = \{a, b, c\}$
- $A_2 = \{\emptyset, \{a, b\}, \top\}$
- $A_3 = \{\emptyset, \{a, c\}, \top\}$
- $\langle \top, \{a, b\}, \{a, c\} \rangle /_{\equiv} = \langle \{a\}, \{a, b\}, \{a, c\} \rangle$
- $\vec{\rho}_{\vec{i}\vec{j}}(\langle \top, \{a, b\}, \{a, c\} \rangle) = \langle \top, \{a, b\}, \{a, c\} \rangle$
  for $\Delta = \{1, 2, 3\}, i, j \in \Delta, i \neq j$
- $\vec{\rho}^{\,*}(\langle \top, \{a, b\}, \{a, c\} \rangle) = \langle \top, \{a, b\}, \{a, c\} \rangle$ is not a minimal element of $[\langle \top, \{a, b\}, \{a, c\} \rangle]/_{\equiv}$

Nelson–Oppen combination procedure

The Nelson-Oppen combination procedure

- Prove satisfiability in a combination of theories by exchanging equalities and disequalities
- Example: $\varphi \triangleq (x = \mathtt{a} \lor x = \mathtt{b}) \land \mathtt{f}(x) \neq \mathtt{f}(\mathtt{a}) \land \mathtt{f}(x) \neq \mathtt{f}(\mathtt{b})$ [22].
  - Purify: introduce auxiliary variables to separate alien terms and put in conjunctive form

$$\begin{aligned}
&\varphi \triangleq \varphi_1 \land \varphi_2 \text{ where} \\
&\varphi_1 \triangleq (x = \mathtt{a} \lor x = \mathtt{b}) \land y = \mathtt{a} \land z = \mathtt{b} \\
&\varphi_2 \triangleq \mathtt{f}(x) \neq \mathtt{f}(y) \land \mathtt{f}(x) \neq \mathtt{f}(z)
\end{aligned}$$

…/…

[22] where a, b and f are in different theories

The Nelson-Oppen combination procedure

$\varphi \triangleq \varphi_1 \wedge \varphi_2$ where
$\varphi_1 \triangleq (x = \mathtt{a} \vee x = \mathtt{b}) \wedge y = \mathtt{a} \wedge z = \mathtt{b}$
$\varphi_2 \triangleq \mathtt{f}(x) \neq \mathtt{f}(y) \wedge \mathtt{f}(x) \neq \mathtt{f}(z)$

- Reduce $\vec{\rho}(\varphi)$: each theory $\mathcal{T}_i$ determines E_{ij} , a (disjunction) of conjunctions of variable (dis)equalities implied by φ_j and propagate it in all other componants φ_i

$E_{12} \triangleq (x = y) \vee (x = z)$
$E_{21} \triangleq (x \neq y) \wedge (x \neq z)$

- Iterate $\vec{\rho}^{\,*}(\varphi)$: until satisfiability is proved in each theory or stabilization of the iterates

The Nelson-Oppen combination procedure

Under appropriate hypotheses (disjointness of the theory signatures, stably-infiniteness/shininess, convexity to avoid disjunctions, etc), the Nelson-Oppen procedure:

- Terminates (finitely many possible (dis)equalities)
- Is sound (meaning-preserving)
- Is complete (always succeeds if formula is satisfiable)
- Similar techniques are used in theorem provers

Program static analysis/verification is undecidable so requiring completeness is useless. Therefore the hypotheses can be lifted, the procedure is then sound and incomplete. No change to SMT solvers is needed.

The Nelson-Oppen procedure is an iterated pairwise reduced product

Observables in Abstract Interpretation

- (Relational) abstractions of values $(v_1,\ldots,v_n)$ of program variables $(\mathtt{x}_1,\ldots,\mathtt{x}_n)$ is often too imprecise.

  Example : when analyzing *quaternions* (a,b,c,d) we need to observe the evolution of $\sqrt{a^2+b^2+c^2+d^2}$ during execution to get a precise analysis of the normalization

- An observable is specified as the value of a function f of the values $(v_1,\ldots,v_n)$ of the program variables $(\mathtt{x}_1,\ldots,\mathtt{x}_n)$ assigned to a fresh auxiliary variable $\mathtt{x}_o$

$$\mathtt{x}_o == f(v_1,\ldots,v_n)$$

(with a precise abstraction of f)

Purification = Observables in A.I.

- The purification phase consists in introducing new observables
- The program can be purified by introducing auxiliary assignments of pure sub-expressions so that forward/backward transformers of purified formulæ always yield purified formulæ
- Example (*f* and *a*,*b* are in different theories):

  y = *f*(x) == *f*(*a*+1) & *f*(x) == *f*(2**b*)

  becomes

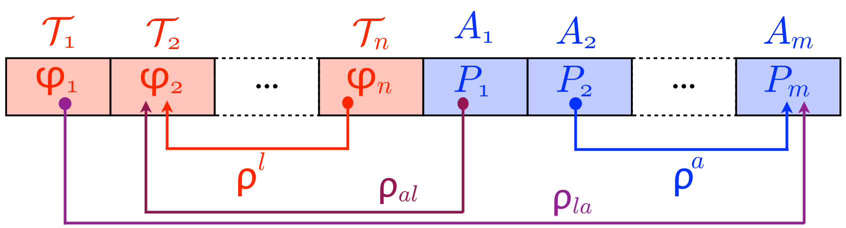
  z=*a*+1; t=2**b*; y = *f*(x) == *f*(z) & *f*(x) = *f*(t)

Reduction

- The transfer of a (disjunction of) conjunctions of variable (dis-)equalities is a pairwise iterated reduction
- This can be *incomplete* when the signatures are not disjoint

Static analysis combining logical and algebraic abstractions

Reduced product of logical and algebraic domains



- When checking satisfiability of $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n$, the Nelson-Oppen procedure generates (dis)-equalities that can be propagated by ρ_{la} to reduce the P_i, $i=1,\ldots,m$, or
- $\alpha_i(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n)$ can be propagated by ρ_{la} to reduce the P_i, $i=1,\ldots,m$
- The purification to theory $\mathcal{T}_i$ of $\gamma_i(P_i)$ can be propagated to φ_i by ρ_{al} in order to reduce it to $\varphi_i \wedge \gamma_i(P_i)$ (in $\mathcal{T}_i$)

Advantages

- No need for completeness hypotheses on theories
- Bidirectional reduction between logical and algebraic abstraction
- No need for end-users to provide inductive invariants (discovered by static analysis)(*)
- Easy interaction with end-user (through logical formulæ)
- Easy introduction of new abstractions on either side

$\Longrightarrow$ Extensible expressive static analyzers / verifiers

(*) may need occasionally to be strengthened by the end-user

Future work

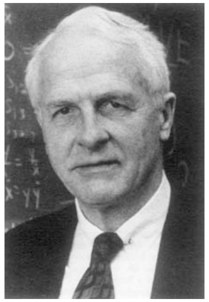
- Still at a conceptual stage
- More experimental work on a prototype is needed to validate the concept

References

1. Patrick Cousot, Radhia Cousot, Laurent Mauborgne: Logical Abstract Domains and Interpretation. In *The Future of Software Engineering*, S. Nanz (Ed.). © Springer 2010, Pages 48—71.
2. Patrick Cousot, Radhia Cousot, Laurent Mauborgne: The Reduced Product of Abstract Domains and the Combination of Decision Procedures. FOSSACS 2011: 456-472

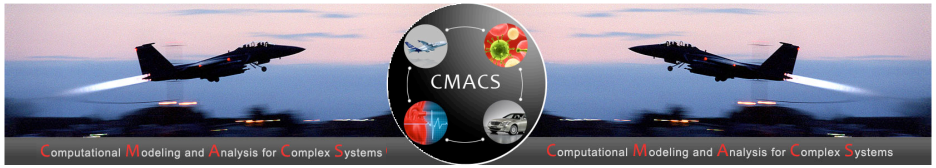
Conclusion

- Convergence between logic-based proof-theoretic deductive methods using SMT solvers/theorem provers and algebraic methods using model-checking/abstract interpretation for infinite-state systems



Garrett Birkhoff (1911–1996)
abstracted *logic/set theory*
into *lattice theory*

1967 (1940). Lattice Theory, 3rd ed. American Mathematical Society.



The End,

Thank You